LVE $(x^2+1)y'' - xy' + y = 0$ USING SERIES.

$$\hookrightarrow y'' - \underbrace{\frac{x}{x^2+1}}_{p} y' + \underbrace{\frac{1}{x^2+1}}_{q} y = 0$$

p, q CONT ON $(-\infty, \infty)$

SO E+U SAYS THERE IS A UNIQUE SOL'N FOR EVERY IVP FROM THIS DE

$$\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$2\cdot 1\cdot a_2 x^0 + 3\cdot 2\cdot a_3 x^1 - 1\cdot a_1 x^1 + a_0 x^0 + a_1 x^1$$

$$+ \sum_{n=2}^{\infty} \left[n(n-1) a_n + (n+2)(n+1) a_{n+2} - n a_n + a_n \right] x^n = 0$$

$$(2a_2 + a_0) + (6a_3 \cancel{- a_1} + a_1)x$$
$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (\underbrace{n^2 - n - n + 1}_{\substack{n^2-2n+1 \\ =(n-1)^2}})a_n \right] x^n = 0$$

$$2a_2 + a_0 = 0 \longrightarrow a_2 = -\tfrac{1}{2}a_0$$

$$6a_3 x = 0x \longrightarrow a_3 = 0$$

$$(n+2)(n+1)a_{n+2} + (n-1)^2 a_n = 0$$

$$a_{n+2} = -\frac{(n-1)^2}{(n+2)(n+1)} a_n \quad \text{FOR } n \geq 2$$

RECURRENCE RELATION
FOR COEFFICIENTS

) LET $a_0 = 1$, $a_1 = 0$

$\hookrightarrow a_1 = 0 = a_3 = a_5 = a_7 = \cdots$

$$a_2 = -\tfrac{1}{2}a_0 = -\tfrac{1}{2}$$

$n=1$

$$= 2: \; a_4 = -\frac{1^2}{4 \cdot 3} a_2 = \frac{1^2}{4 \cdot 3 \cdot 2}$$

$n=2$

$$a_{2n} = (-1)^n \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2}{(2n)!}$$

$n=3$

$$= 4: \; a_6 = -\frac{3^2}{6 \cdot 5} a_4 = -\frac{3^2 \cdot 1^2}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

CAN BE SIMPLIFIED

$n=4$

$$= 6: \; a_8 = -\frac{5^2}{8 \cdot 7} a_6 = \frac{5^2 \cdot 3^2 \cdot 1^2}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\underbrace{1\cdot 3\cdot 5\cdots(2n-3)}_{\#\text{ FACTORS}} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots(2n-3)(2n-2)}{2\quad 4\quad 6 \qquad (2n-2)}$$

$$= \frac{(2n-2)!}{2^{n-1}(1\cdot 2\cdot 3\cdots(n-1))} = \frac{(2n-2)!}{2^{n-1}(n-1)!}$$

$$\underbrace{1\cdot 3\cdot 5\cdots(2n-3)}_{\#\text{ FACTORS}}\underbrace{(2n-1)}_{+\ 1} = \frac{(2n-1)(2n-2)!}{2^{n-1}(n-1)!}$$

$$= \frac{(2n-1)!}{2^{n-1}(n-1)!}\,\frac{2n}{2n}$$

$$= \frac{(2n)!}{2^n\cdot n!}$$

SUB $n \longrightarrow n+1$

$$\frac{(2(n+1)-2)!}{2^{(n+1)-1}((n+1)-1)!}$$

$$\frac{(2n)!}{2^n\, n!}$$

$$a_{2n} = (-1)^n \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2}{(2n)!}$$

$$= (-1)^n \frac{\left(\frac{(2n-2)!}{2^{n-1}(n-1)!}\right)^2}{(2n)!}$$

$$= \frac{(-1)^n [(2n-2)!]^2}{2^{2n-2}[(n-1)!]^2(2n)!}$$

IF $n=1$ $\frac{(-1)^1 (0!)^2}{2^0 (0!)^2 2!} = -\frac{1}{2} = a_2$

$$\sum_{n=1}^{\infty} a_{2n} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n [(2n-2)!]^2}{2^{2n-2}[(n-1)!]^2(2n)!} x^{2n} = y_1$$

) IF $a_0 = 0, a_1 = 1$

$\hookrightarrow a_2 = a_4 = a_6 = \cdots = 0$

$$a_{n+2} = \frac{-(n-1)^2}{(n+2)(n+1)} a_n \text{ FOR } n \geq 2$$

$a_3 = 0 \rightarrow a_5 = a_7 = a_9 = \cdots = 0$

~~n=3: $a_5 =$~~

$y_2 = 1x = x$ NOTE: $y_2' = 1, y_2'' = 0$

$(x^2+1)(0) - x(1) + x = 0$ ✓

$y = C_1 y_1 + C_2 x$

METHOD OF FROBENIUS

$$y'' + p(x)y' + q(x)y = 0$$ WHAT IF p, q ARE NOT BOTH CONT?

IF $p(x), q(x)$ ARE BOTH CONT. @ $x = x_0$, WE SAY $x = x_0$ IS AN ORDINARY POINT OF DE

IF EITHER $p(x)$ OR $q(x)$ OR BOTH ARE DISCONT, @ $x = x_0$, WE SAY $x = x_0$ IS A SINGULAR POINT OF DE

(WE WILL ONLY CONSIDER IF $x = 0$ IS A SINGULAR POINT OF DE)

eg. $$Ax^2y'' + Bxy' + Cy = 0$$

$$y'' + \frac{B}{Ax}y' + \frac{C}{Ax^2}y = 0$$ E+U DOESN'T APPLY AT $x = 0$

$x = 0$ IS A SINGULAR POINT OF ALL CAUCHY-EULER EQ'NS

$$y'' + p(x)y' + q(x)y = 0$$

IF EITHER p OR q ARE DISCONT. @ $X = X_0$

[AND $\lim_{x \to x_0} (x-x_0)p(x)$ AND $\lim_{x \to x_0} (x-x_0)^2 q(x)$ BOTH EXIST]

AND $(x-x_0)p(x)$ AND $(x-x_0)^2 q(x)$ ARE BOTH $\underbrace{\text{ANALYTIC}}$

THEN WE SAY $X = X_0$ IS A REGULAR SINGULAR POINT

INFINITELY DIFFERENTIABLE AT $X = X_0$

eg. $Ax^2y'' + Bxy' + Cy = 0$ $\quad X = 0$ IS A SINGULAR POINT

$$y'' + \frac{B}{Ax}y' + \frac{C}{Ax^2}y = 0$$

$$\left.\begin{array}{ll} p(x) = \frac{B}{Ax} & xp(x) = \frac{B}{A} \\ q(x) = \frac{C}{Ax^2} & x^2 q(x) = \frac{C}{A} \end{array}\right\} \text{BOTH ANALYTIC AT } X = 0$$

SO $X = 0$ IS A REGULAR SINGULAR POINT.

TRY $$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

METHOD OF FROBENIUS

VE $2xy'' + 5y' + xy = 0$ USING A MACLAURIN SERIES

↳ ABOUT $x=0$

(IS A SINGULAR POINT OF DE)

$$\downarrow$$

$$y'' + \frac{5}{2x}y' + \frac{1}{2}y = 0$$

$$\left.\begin{aligned} p(x) &= \frac{5}{2x} \qquad xp(x) = \frac{5}{2} \\ q(x) &= \frac{1}{2} \qquad x^2 q(x) = \frac{1}{2}x^2 \end{aligned}\right\}$$ BOTH ANALYTIC @ $x=0$ $\longrightarrow$ $x=0$ IS A REGULAR SINGULAR POINT

LET $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + 5y' + xy = 2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$+ 5 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ x \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1}$$ SHIFT DOWN BY 2

$$= \quad '' \quad + \quad '' \quad + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

$$= 2r(r-1)a_0 x^{r-1} + 2(r+1)(r)a_1 x^r$$

$$+ 5r a_0 x^{r-1} \qquad + 5(r+1)a_1 x^r$$

$$+ \sum_{n=2}^{\infty} \left[2(n+r)(n+r-1)a_n + 5(n+r)a_n + a_{n-2}\right] x^{n+r-1}$$

$$= (2r(r-1)+5r)a_0 x^{r-1} + (2(r+1)r + 5(r+1))a_1 x^r$$

$$+ \sum$$